Lecture 19

9.3 - Separable Equations

Suppose we have a differential equation

$$y' = f(x,y)$$

which can be written in the form

$$p(y)\,y' = q(x)$$

Now, if $y = g(x)$ is a solution of this, then

$$p(g(x))\,g'(x) = q(x) \qquad (*)$$

and we can integrate with respect to x on both sides (using the u-sub $u = g(x)$ on the left side).
If $P(y)$ & $Q(x)$ are functions such that $P'(y) = p(y)$ and $Q'(x) = q(x)$, then integrating $(*)$ gives:

$$P(g(x)) = Q(x) + C$$

Replacing $g(x)$ with y again, we see

$$P(y) = Q(x) + C. \qquad (**)$$

So a solution of $y' = f(x,y)$ should satisfy $(**)$.

Conversely, differentiating (**) gives

$$P'(y)y' = Q'(x)$$

$$\Leftrightarrow p(y)y' = q(x)$$

So, this method indeed gives solutions to $y' = f(x,y)$.

Differential equations of this type are called <u>separable</u>.

Writing $y' = \frac{dy}{dx}$, we can rewrite

$$p(y)y' = q(x) \Leftrightarrow p(y)\frac{dy}{dx} = q(x) \Leftrightarrow \boxed{p(y)\,dy = q(x)\,dx}$$

Sometimes it isn't always possible to solve (**); in these cases, we leave the equation as is. These are called <u>implicit solutions</u>.

<u>Ex</u>: Solve $y' = -2xy^2$.

<u>Sol</u>: $\frac{1}{y^2}\,dy = -2x\,dx$ (need to check $y=0$!)

<u>If $y=0$</u>: $\underbrace{\frac{d}{dx}(0)}_{0} = \underbrace{-2x(0)^2}_{0}$ ✓

$y=0$ is a solution.

<u>Now, assume $y \neq 0$</u>:

<u>Integrate</u>:

$$-\frac{1}{y} = \int \frac{1}{y^2}\,dy = \int -2x\,dx = -x^2 + C$$

$$\Rightarrow y = \frac{1}{x^2+C}.$$

Since no C value makes $y=0$, the solutions are: $\boxed{y = \frac{1}{x^2+C},\ y=0}$

Ex: Solve $y'-2xy=6x$.

Sol: $y'-2xy=6x \iff y'=6x+2xy=2x(3+y)$

$\iff \frac{1}{3+y}\,dy=2x\,dx$ ⚠ Need to check $y=-3$

If $y=-3$: $\frac{d}{dx}(3)-2x(-3)=6x$ ✓

$\phantom{\frac{d}{dx}(3)}\;\; \overset{\parallel}{0+6x}$ So, $y=-3$ is a solution!

$y\neq -3$: $\int \frac{1}{3+y}\,dy=\int 2x\,dx \Rightarrow \ln|3+y|=x^2+C$

Exponentiate both sides: $|3+y|=e^{x^2+C}=e^Ce^{x^2}$

Remove abs. val: $3+y=\pm e^Ce^{x^2} \iff y=\pm e^Ce^{x^2}-3$

$\pm e^C$ can be anything but zero. We'd like to replace it with just C, but we need to make sure $C=0$ is okay. If we write $y=Ce^{x^2}-3$ and let $C=0$, we get $y=-3$, which is a solution. So, we're okay. Thus the solution is $\boxed{y=Ce^{x^2}-3}$

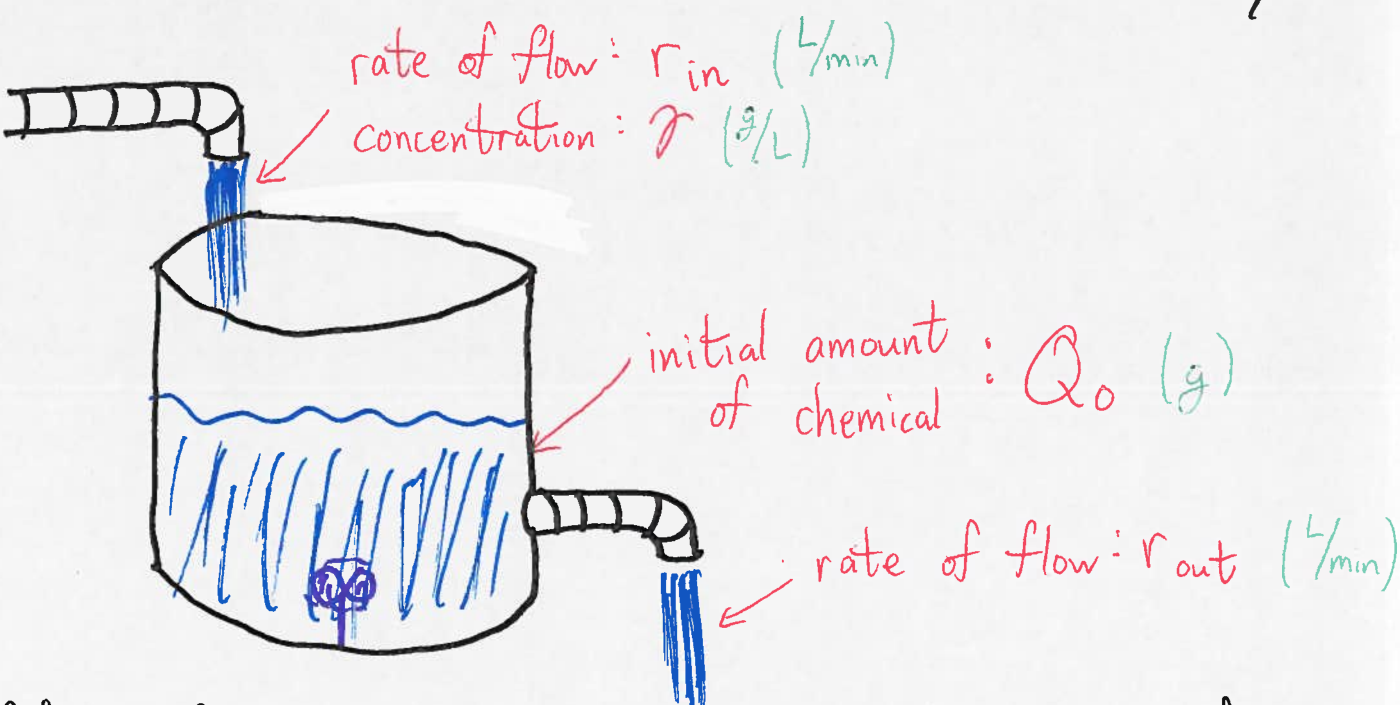
In the case of an IVP $p(y)\,dy=q(x)\,dx$, $y(x_0)=y_0$, a neat trick to get the solution is, when you integrate do so from the initial value to the variable, i.e., the solution is given by

$$\boxed{\int_{y_0}^{y} p(s)\,ds=\int_{x_0}^{x} q(t)\,dt}$$

Some Applications

Mixing Problems

Suppose we have a tank holding a chemical solution and that we are pumping in this same chemical solution at a given rate and with a given concentration. Further, assume that we know what rate the solution is leaving the tank and that the well-mixed solution is leaving. Using this information, we can find the amount of the chemical in the tank at any time.



Using this info, we set up an IVP describing the amount of chemical in the tank at any time t: $Q(t)$

$$Q'(t) = \text{rate in of chemical } \tfrac{g}{min} - \text{rate out of chemical } \tfrac{g}{min}$$

$$= (r_{in})(\gamma) - (r_{out})\left(\frac{Q(t)}{V(t)}\right)$$

$$Q(0) = Q_0$$

$V(t)$ = volume of solution in tank $= \underbrace{V_0}_{\text{initial volume}} + \underbrace{(r_{in} - r_{out})}_{\text{net rate of change in volume}} t$

<u>Ex</u>: A tank originally contains 100 L of fresh water. An alcohol solution with concentration 5% per liter flows into the tank at a rate of 10 liters per minute. The solution leaves the tank at the same rate. What is the concentration of alcohol in the tank after one hour?

<u>Sol</u>: Let's identify the relevant constants:

$r_{in} = r_{out} = 10 \ L/min$, $\gamma = \frac{1}{20} \frac{\text{L of alcohol}}{\text{L of solution}}$

Fresh water: $Q(0) = 0$

$V(t) = 100 + (10-10)t = 100$

$$\Rightarrow \begin{cases} Q'(t) = (10)\left(\frac{1}{20}\right) - (10)\left(\frac{Q(t)}{100}\right) = \frac{1}{2} - \frac{1}{10}Q(t) \\ Q(0) = 0 \end{cases}$$

$$Q' = \frac{1}{2} - \frac{1}{10}Q = \frac{1}{2}\left(1 - \frac{1}{5}Q\right) \Rightarrow \frac{1}{1-\frac{1}{5}Q}\,dQ = \frac{1}{2}dt$$

Integrate: $\ln\left|1 - \frac{Q}{5}\right| = \frac{1}{2}t + C \Leftrightarrow \left|1 - \frac{Q}{5}\right| = e^{\frac{1}{2}t + C} = e^{C}e^{t/2}$

$$\Rightarrow 1 - \frac{Q}{5} = \pm e^{C}e^{t/2} \Rightarrow Q = 5\left(1 \pm e^{C}e^{t/2}\right)$$

Initial value: $Q(0) = 5(1 \pm e^{C}) = 0 \Rightarrow \pm e^{C} = -1.$

Solution: $\boxed{Q(t) = 5(1 - e^{t/2})}$ | After one hour $(t=60)$: $\boxed{Q(60) = 5(1 - e^{30}) \text{ L of alcohol}}$

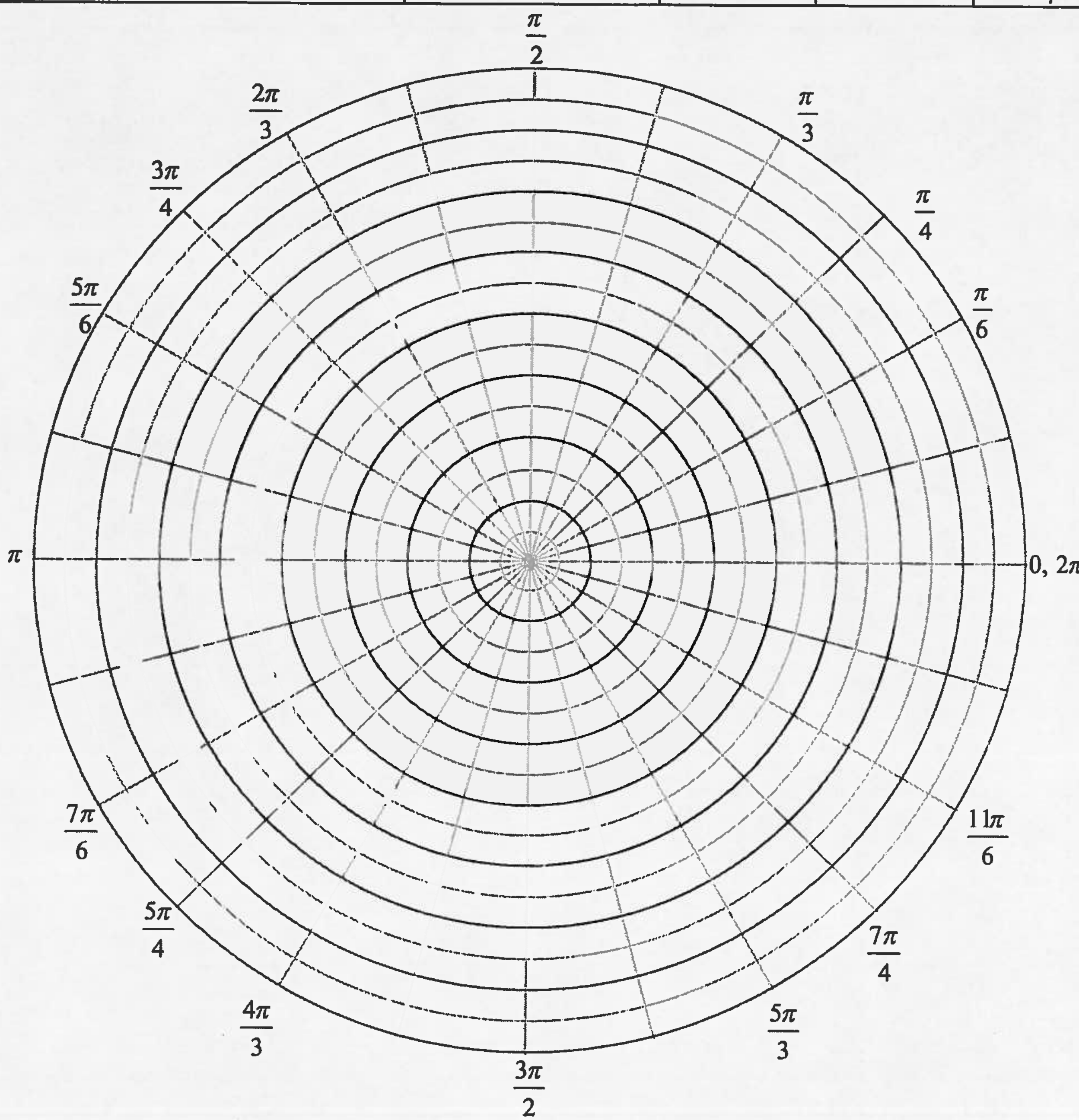
Orthogonal Trajectories

Given a family of curves, an orthogonal trajectory to the family is a curve that intersects every member of the family at right angles, i.e., their slopes at the points of intersection multiply to be -1 (i.e., they are negative reciprocals of each other). Since slopes are given by derivatives, given a family of curves, we can use differential equations to find orthogonal trajectories.

A simple example is the family of lines through the origin $(y = kx \;\&\; x = 0)$ is orthogonal to the family of concentric circles centered at the origin $(x^2 + y^2 = a^2)$.

See picture:

Name: Description:	Section/Page Numbers:	Problems:	Date:	Page 1



Ex: Find the orthogonal trajectories to the family of parabolas $y=ax^2$. Plot a few from each family.

$a=\frac{y}{x^2}$

Sol: $y'=2ax \xrightarrow[\text{slopes}]{\text{orthogonal}} y'=\frac{-1}{2ax}=\frac{-x}{2y}$

$$\Rightarrow \int 2y\,dy=\int x\,dx \Rightarrow y^2=-\frac{1}{2}x^2+C$$

$$\Rightarrow x^2+2y^2=C$$: family of ellipses